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Theory of Ragone plots

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Abstract

The general theory of Ragone plots for energy storage devices (ESD) is discussed. Ragone plots provide the available energy of an ESD for constant active power request. The qualitative form of Ragone plots strongly depends on the type of storage (battery, capacitor, SMES, flywheel, etc.). For example, the energy decreases as a function of power for capacitive ESD, but increases for inductive ESD. Analytical results for a representative set of ideal ESD (battery, capacitor, and SMES) are compared. Furthermore, the effect of leakage and of the specific loss type (Coulomb, Stokes, and Newton friction) is discussed for inductive ESD. Finally, we address the problem of how composite ESD should be treated, and illustrate it for a battery with inductance.

Keywords: Energy–power relations; Batteries; Capacitors; SMES

1. Introduction

Energy storage devices (ESD) are characterized by the energy and the power being available for a load [1,2]. A prominent example is the comparison of conventional batteries and capacitors. While batteries have high energy densities (about 10^5J/kg specific energy) but only low power densities (below 100 W/kg specific power), capacitors have rather high power densities (about 10^6 W/kg) but low energy densities (about 100 J/kg). Batteries, capacitors, flywheels, superconducting magnetic energy storage devices (SMES), pressure storage devices, etc., are thus located in characteristic regions in the power–energy plane. Typical examples are shown in Fig. 1. These regions are related to specific applications by energy and power requirements. The boundaries of the regions are determined by internal losses and/or leakage, etc., of the various ESD. The characteristic time of an application is of the order of the energy-to-power ratio of the ESD. In the log–log plane of Fig. 1, the time corresponds to straight lines. Obviously, batteries are useful for long time applications (>100 s), while conventional capacitors are useful for short time applications (<0.01 s).

Since the efficiency of an ESD is usually dependent on the working point, a single device belongs to a whole curve in the energy–power plane (see inset of Fig. 1). These so-called Ragone plots , which are usually presented in a log–log plot, are standard in the battery community since a long time [1]. First, they provide the limit in the available power of a battery or a capacitor. Secondly, they provide the optimum region of working, which is given by the part of the curve where both energy and power are high. The aim of this paper is to present a unified discussion of the qualitative behavior of Ragone plots for different ESD. Here, we will focus on the specific curves rather than on the specific regions where these curves are located. It turns out also that the specific form of the Ragone curve depends on internal loss and leakage properties of the ESD. A typical qualitative behavior of a Ragone curve is sketched in the inset of Fig. 1. Consider for example a capacitor or a battery. The internal self discharge leads to a decrease of the energy that can be utilized, if the characteristic time of the application exceeds the self discharge time. This fact corresponds to a drop of the Ragone curve for sufficiently low power. On the other hand, the effective series resistance leads to a lower time limit and thus to a maximum power. It is clear that, irrespective of the type of ESD, there are always physical limits to minimum and

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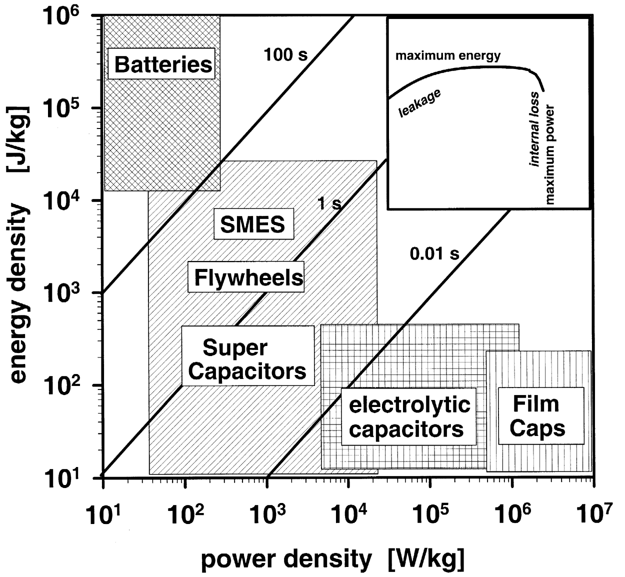

Fig. 1. Ragone plane: available energy of an energy storage device for fixed power. Different types of energy storage devices are typically located in different regions. Characteristic times correspond to lines with unity slope. Every energy storage device is represented by a curve $E(P)$ (inset). Internal dissipation and leakage losses lead to a drop of the energy for sufficiently high and low power.

maximum speed of discharge of an ESD. These limits are reflected in the low/high power behavior of the Ragone plot.

In the next section, we introduce the general class of ESD that will be investigated, and we propose a mathematical definition of the Ragone plot. In Section 3, we discuss two specific cases of potential ESD, which may be interpreted as ideal battery and capacitor. In Section 4, the Ragone plot of a purely inductive ESD is studied, which may be interpreted as a SMES or a flywheel. Furthermore, the effect of various types of friction forces in kinetic ESD is addressed. Section 5 provides a brief discussion of the stability problem of circuits containing a constant power load; as a particular example, we discuss the battery with a series inductance.

2. Ragone plots

Consider the general circuit of Fig. 2. For example, the ESD may consist of a voltage source, $V(Q)$, depending on the stored charge Q, an internal series resistor, R, and an internal inductance, L. Note that this ESD can describe many kinds of electrical power sources. For example, a current source delivering a current I_0 can be described by $L=0$, and R, $V\to\infty$, with $V/R\to I_0$. The ESD is connected to a load which draws constant (active) power $P\geq 0$. Of course, in general such a load is not related to a constant resistance (except for the battery), but requires control engineering. We assume first that the load has no reactive power. (In Section 5, we show that in general an additional apparent power must be considered.) The current I and voltage U at the load are then related nonlinearly by $U=P/I$. Provided reasonable initial conditions, $Q(0)=Q_0$ and $\dot{Q}(0)=\dot{Q}_0$, are given, the electrical dynamics is governed by the following ordinary differential equation (ODE) for $Q(t)$,

$$L\ddot{Q}+R\dot{Q}+V(Q)=-\frac{P}{\dot{Q}}, \qquad (1)$$

where the dot indicates differentiation with respect to time. This equation applies not only to electrical ESD but covers many kinds of physical systems (mechanical, hydraulic, etc.). For example, identifying L, R, and $V(Q)=\mathrm{d}W/\mathrm{d}Q$ with inertia (mass), Stokes friction constant, and a (negating) force, respectively, the left hand side of Eq. (1) is Newton's equation for the coordinate Q of a mechanical particle in the potential W. The right-hand side describes a force acting on the particle and leading to an energy release with power P. A similar equivalence holds for flywheels where Q denotes the angle coordinate.

Even without reference to a specific physical interpretation of Eq. (1), we can define the Ragone curve as follows. At time $t=0$, the device contains the stored energy, $E_0=L\dot{Q}_0^2/2+W(Q_0)$. For $t\geq 0$, the load draws a constant power P such that $Q(t)$ satisfies Eq. (1). It is clear that, for finite E_0 and P, the ESD is able to supply this power only for a finite time, say $t=t_\infty(P)$. A criterion is given either by the time when the storage device is cleared or when the ESD is no longer able to deliver the required amount of power. Since the power P is time independent, the available energy is $E(P)=Pt_\infty(P)$. The curve $E(P)$ versus P corresponds to the Ragone plot. This approach is in fact independent of the specific ESD, which is the reason why we call it 'general theory' of Ragone plots. Note that our definition is different from the definition discussed by Pell and Conway [3,4]. These authors consider the energy of the ESD but not $Pt_\infty(P)$.

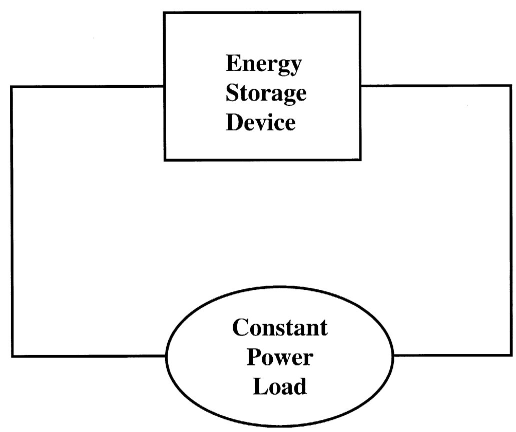

Fig. 2. General circuit associated with Ragone plots: the energy storage device (ESD) feeds a load with constant (active) power consumption P. The ESD contains elements for energy storage. Due to constant power, energy supply occurs only for a finite time, $t_\infty(P)$. The energy available for the load, $E(P)=Pt_\infty$, defines a Ragone plot.

Before we calculate $E(P)$ for a few idealized cases, we mention the trivial case without internal loss, $R=0$. It is clear that for any reasonable ESD the full energy is then available and $t_\infty = E_0/P$, such that the Ragone curve $E(P)=E_0$ is constant for all $P \geq 0$. The reader can easily convince himself that the results below will reproduce this behavior in the limit of vanishing losses and leakage.

3. Storage of potential energy

In this section, we consider ESD without inductance ($L=0$). We focus on the particular cases of an ideal battery and an ideal capacitor. 'Ideal' means that there is neither frequency dependence nor an intrinsic nonlinearity (e.g. faradaic contributions, pseudo-capacitance, etc.). Battery and capacitor differ in their charge dependence of $V(Q)$ in Eq. (1).

3.1. Battery

The ideal battery with capacity Q_0 (inset Fig. 3) is characterized here by a constant (charge independent) reversible cell voltage, $V=U_0$ if $Q_0 \geq Q > 0$ and $V=0$ if $Q=0$. In a first step, we disregard the leakage resistance R_L. Eq. (1) reads $P=UI=(U_0-RI)I$, where U is the terminal voltage and $I=\dot{Q}$ is the current. The solutions of this quadratic equation are

$$I_\pm = \frac{U_0}{2R} \pm \sqrt{\frac{U_0^2}{4R^2} - \frac{P}{R}}\,. \qquad (2)$$

In the limit $P \to 0$, the two branches correspond to a discharge current $I_+ \to U_0/R$ and to $I_- \to 0$. For the ideal battery, the constant power sink can also be parameterized by constant load resistance, R_{Load}. The two limits belong then to $R_{Load} \to 0$ (short circuit) and $R_{Load} \to \infty$ (open switch), respectively. Clearly, in the context of the Ragone plot, we are interested in the latter limit, such that we have to take the branch with the minus sign, $I \equiv I_-$, in Eq. (2).

Now, the battery is empty at time $t_\infty = Q_0/I$, where the initial charge Q_0 is related to the initial energy by $E_0 = Q_0U_0$. It is now easy to include the presence of an ohmic leakage current into the discussion. The leakage resistance R_L increases the discharge current I by U_0/R_L. The energy being available for the load becomes

$$E_b(P) = Pt_\infty = \frac{2RQ_0P}{U_0 - \sqrt{U_0^2 - 4RP} + 2U_0R/R_L}\,. \qquad (3)$$

Eq.(3) corresponds to the Ragone curve of the ideal battery. In the presence of leakage, $E_b(0)=0$ and there exists a maximum at $P \approx U_0^2/\sqrt{2RR_L}$. Without leakage ($R/R_L \to 0$), the maximum energy is available for vanishingly low power, $E_b(P\to 0)=E_0$. From Eq. (3), one concludes that there is a maximum power, $P_{max}=U_0^2/4R$, associated with an energy $E_0/2$ (here, we neglected a small correction due to leakage). This point is the endpoint of the Ragone curve of the ideal battery, where only half of the energy is available while the other half is lost at the internal resistance.

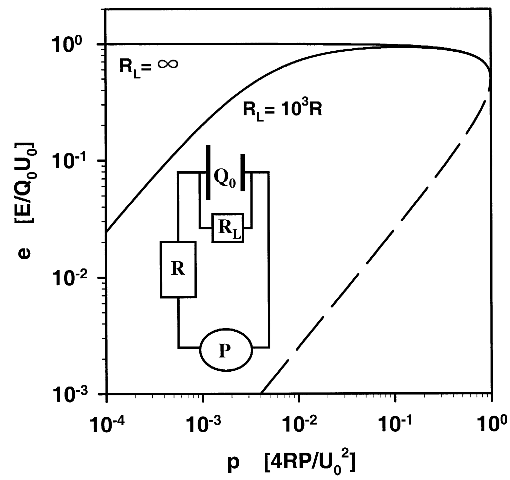

Fig. 3. Solid curves: Ragone plots of an ideal battery with and without leakage (resistance R_L). Dashed curve: secondary branch of the energy–power relation, not useful for a Ragone curve (see text). Inset: constant power load P connected to a battery with capacity Q_0, ESR R, and leakage resistance R_L.

Let us finally express the Ragone plot for the battery in the dimensionless units $e_b = E_b/Q_0U_0$ and $p = 4RP/U_0^2$

$$e_b(p) = \frac{1}{2}\,\frac{p}{1-\sqrt{1-p}+2R/R_L}\,. \qquad (4)$$

Ragone curves (Eq. (4)) with and without leakage are shown in Fig. 3. The branch belonging to I_+ is plotted by the dashed curve. A more detailed description of batteries including Tafel polarization, concentration polarization, etc., is discussed in a different way by Pell and Conway [3].

3.2. Capacitor

The case of an ideal electric capacitor (inset Fig. 4) is a little more laborious than the case of the ideal battery, since now an ODE rather than an algebraic equation has to be solved. The electric potential $V(Q)=Q/C$ depends linearly on the charge via a capacitance C. It is convenient to derive from Eq. (1) an equation for the voltage drop at the load, $U=P/I=Q/C-RI$ with $I=-\dot{Q}$. Differentiating both expressions for U with respect to time, replacing $\dot{I}$ in the first expression with the help of the second one, and multiplying by U leads to the following ODE for U^2

$$\left(1-\frac{RP}{U^2}\right)\frac{dU^2}{dt} = -\frac{2P}{C}\,. \qquad (5)$$

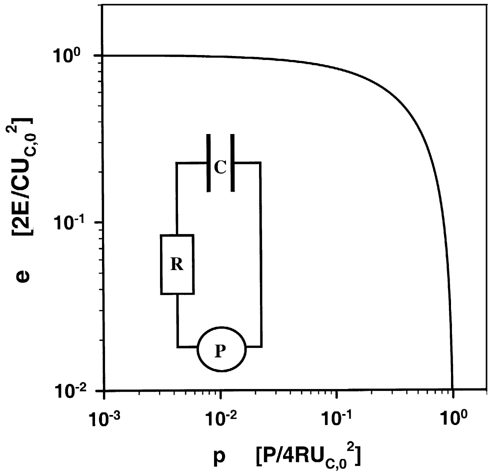

Fig. 4. Normalized Ragone curve for the capacitor. Inset: constant power load P connected to a capacitor with capacitance C and ESR R.

Separation of variables and integration gives the solution

$$t(U) = \frac{C}{2P}\left(RP\ln\left(\frac{U^2}{U_0^2}\right) + U_0^2 - U^2\right). \tag{6}$$

It turns out that there exists a turning point with $\mathrm{d}t/\mathrm{d}U = 0$, where the capacitor is no longer able to supply the required power P. One finds for the corresponding value of the voltage $U_\infty = \sqrt{RP}$ (assuming positive U). Note that in contrast to the battery, the capacitor is not empty at $t_\infty = t(U_\infty)$, but there is a residual energy $E_\infty = 2RCP$. This relation already anticipates the existence of a maximum power $P_{\max} = E_0/2RC$. The time t_∞ is reached when the capacitor is no longer able to supply the required power P. In order to calculate the Ragone plot, $E_c = Pt_\infty$, one has to be careful. Indeed, U_0 in Eq. (6) depends on P, because $U_C = U + RI = U + RP/U$ is the voltage drop at the capacitance. Hence, the Ragone curve of the capacitor is

$$E_c(P) = \frac{C}{2}\left(RP\ln\left(\frac{RP}{U_0^2}\right) + U_0^2 - RP\right), \tag{7}$$

$$U_0 = \frac{U_{C,0}}{2} + \sqrt{\frac{U_{C,0}^2}{4} - RP}\,, \tag{8}$$

where the initial capacitor voltage $U_{C,0}$ is related to the total energy by $E_0 = CU_{C,0}^2/2$. Note, as for the battery, we had to choose $U_{C,0}$ out of two branches. However, we will not discuss the secondary branch, which is irrelevant for our purposes. As mentioned above, there exists a maximum power $P_{\max} = U_{C,0}^2/4R$ that can be delivered by the capacitor and for which $E_c \to 0$. Our results differ from those of Pell and Conway [3], who find a finite energy for $P \to P_{\max}$. The reason lies in the different definitions of the Ragone curves, $E(P)$. For vanishing power, $P \to 0$, the whole energy E_0 is available. In the presence of leakage, it holds $E_c(P) \to 0$ for $P \to 0$, as for the battery. However, this case will not be discussed here, since the discharge time of a capacitor is much shorter than usual leakage times. In the dimensionless units $e_c = 2E_c/CU_{C,0}^2$ and $p = 2RCP/E_0$, the Ragone curve reads

$$e_c(p) = \frac{1}{4}\left(\left(1+\sqrt{1-p}\right)^2 - p \right. \tag{9}$$

$$\left. - p\ln\left(\frac{\left(1+\sqrt{1-p}\right)^2}{p}\right)\right). \tag{10}$$

This result is shown in Fig. 4.

4. Storage of kinetic energy

Inductive or kinetic ESD store energy exclusively as kinetic energy (coming from the L-term of Eq. (1)), i.e., $V \equiv 0$. In contrast to storage of potential energy, where load-free losses ($P \to 0$) are related to a separate 'leakage' property, the kinetic ESD dissipates energy due to internal friction, related to R in Eq. (1). In practice, inductive ESD have thus very small friction forces or internal resistances. We will show that internal friction influences kinetic ESD only at low power, and that the qualitative behavior of the Ragone plot is insensitive to the specific friction among the most common types. At high power, the energy turns out to be limited by a finite bypass resistance.

4.1. SMES

In the following, we consider a SMES where energy is stored inductively, i.e., in the magnetic field of the inductance L (inset Fig. 5). Nevertheless, the following discussion holds also for other kinds of inductive storage de-

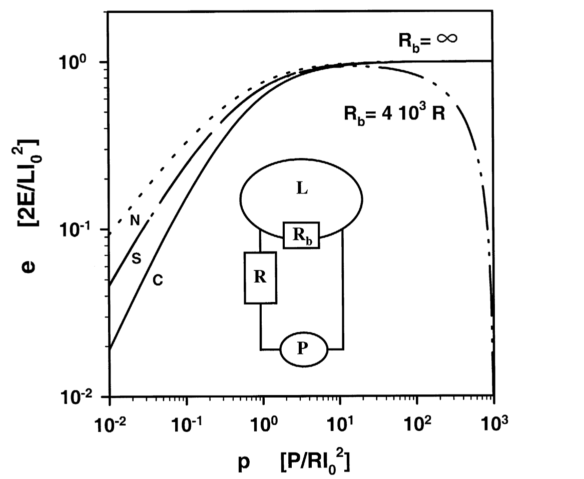

Fig. 5. Normalized Ragone curves for the inductive ESD with Coulomb (C), Stokes (S), and Newton (N) friction. The dashed double-dotted curve corresponds to a SMES with an ohmic bypass, $4R/R_b = 0.001$. Inset: constant power load connected to a SMES with inductance L, ESR R, and bypass resistance R_b.

vices. The stored energy is $E_0 = LI_0^2/2$, where L is the inductance and I_0 is the initial loop current. At time $t = 0$, the loop is opened by inserting a high resistance R_b and immediately closed by connecting it in series to the constant power load.[1] The series resistance R consists of all losses in contacts, switches, etc., except the losses in the load. Consider first the case where the bypass resistance can be disregarded, $R_b \to \infty$. The basic Eq. (1) reduces to a first-order ODE for the current, $L\mathrm{d}I/\mathrm{d}t + RI = -P/I$. (By the way, the mechanical interpretation corresponds to a deceleration of a moving particle with velocity I at constant power gain.) Multiplication with I leads to the following ODE for the kinetic energy, $W_{kin} = LI^2/2$,

$$\frac{\mathrm{d}W_{kin}}{\mathrm{d}t} + \frac{W_{kin}}{\tau_L} + P = 0, \tag{11}$$

where $\tau_L = L/2R$ is the energy relaxation time. The solution of Eq. (11) with initial condition $W_{kin}(0) = E_0$ is

$$W_{kin}(t) = \frac{L}{2}\left(I_0^2 + \frac{P}{R}\right)e^{-t/\tau_L} - \frac{PL}{2R}. \tag{12}$$

In the inductive case, there is only a single solution branch. The storage device is empty when $W_{kin}(t) = 0$. Solving this for $t(= t_\infty(P))$ yields for the Ragone curve

$$E_i(P) = t_\infty P = \frac{PL}{2R}\ln\left(1 + \frac{RI_0^2}{P}\right). \tag{13}$$

The ideal inductive ESD show a behavior fully different from an ideal device storing potential energy without leakage. In particular, $E_i \to 0$ for $P \to 0$, and $E_i \to E_0$ for $P \to \infty$. The total energy is available for fast energy consumption, while for low power everything is lost in R. This becomes clear in the framework of the particle picture. Drawing energy from a moving particle requires the action of a friction force, which in turn diminishes the particle's velocity. If this friction force is much weaker than internal friction, the total energy is dissipated internally and only a small fraction of the energy is available for the constant power load.

In reality it is of course not possible to get arbitrarily large power. Cutoffs must exist for all kinds of ESD and for both limits $P \to 0$ and $P \to \infty$. Similar to the leakage resistance of a battery, which becomes important at $P \to 0$, for a SMES the bypass resistance R_b (e.g., via the power electronics) must be taken into account for $P \to \infty$. This means that the load current I_P is the total current I minus the bypass current I_b. Since the voltage across the load and the bypass is given by $P/I_P = R_b I_b$, one finds for the load current $I_P = (I + \sqrt{I^2 - 4P/R_b})/2$. Consequently, the differential equation for the SMES current becomes

$$L\dot{I} = -RI - \frac{2P}{I + \sqrt{I^2 - 4P/R_b}}. \tag{14}$$

It is clear from Eq. (14) that there exists a maximum power value $P_{max} = R_b I_0^2/4$. Eq. (14) is solved by first transformation to the variable W_{kin}, and then separation of the variables t and W. One ends with an integral expression for $t(W)$, which leads to $t_\infty(P)$ by taking $W = 2PL/R_b$. The result for the Ragone plot, $E = t_\infty(P)$, will be presented in dimensionless variables in the following sub-section.

4.2. Dependence on the type of friction

Three different types of friction forces are often observed in physical systems. First, Coulomb friction is velocity independent, $F_{fr} = -\mathrm{sign}(\dot{Q})K_C$, and is observed in many mechanical systems. Secondly, Stokes friction is proportional to the velocity, $F_{fr} = -K_S\dot{Q}$, and occurs usually in viscous flow. Thirdly, Newton friction is proportional to the kinetic energy, $F_{fr} = -K_N \cdot \mathrm{sign}(\dot{Q})\dot{Q}^2$, and occurs in weakly turbulent flow. Dissipation in electric systems corresponds to Stokes friction with $K_S = R$, according to the fact that a current can be understood as a viscous flow of charge carriers. In an inductive ESD, Stokes friction gives the Ragone curve (Eq. (13)), which can be expressed in the dimensionless units $e_S = 2E_i/LI_0^2$ and $p = P/RI_0^2$ as

$$e_S(p) = p\ln\left(1 + \frac{1}{p}\right). \tag{15}$$

In the presence of a finite bypass resistance, the discussion at the end of the previous subsection leads to

$$e_S(p) = p\int_{\frac{4pR}{R_b}}^{1} \frac{\mathrm{d}w}{w + \dfrac{2p}{1 + \sqrt{1 - \dfrac{4Rp}{R_b w}}}}. \tag{16}$$

The result is shown in Fig. 5 by the dashed double-dotted curve. One clearly observes a sharp drop at high power $P \approx R_b I_0^2$.

In the following, we compare the result of Eq. (15) with the Ragone plots for the kinetic systems with Coulomb and Newton friction.

Replacing $R\dot{Q}$ by the Coulomb friction constant, K_C, in Eq. (1) with $V \equiv 0$, leads to the ODE $L\dot{I} + K_C = -P/I$. Integration of this ODE and putting $I(t_\infty) = 0$ yields the Ragone curve $E_C(P) = Pt_\infty$. In dimensionless units $e_C = E_C/E_0$, $p = P/K_C I_0$, the Ragone plot reads

$$e_C(p) = 2p\left(1 - p\ln\left(1 + \frac{1}{p}\right)\right). \tag{17}$$

[1] We mention that the switch problem is technically non-trivial; however, here we are mainly interested in the principal response of an inductive ESD.

An analogous calculation leads to the Ragone curve $E_N(P)$ for Newton friction, which reads in the dimensionless units $e_N = E_N/E_0$ and $p = P/K_N I_0^3$ as follows

$$e_N(p) = \frac{2p^{2/3}}{3}\left(\sqrt{3}\left(\arctan\{2p^{-1/3}/\sqrt{3} - 1/\sqrt{3}\}\right.\right. \quad (18)$$

$$\left.\left. + \pi/6\right) + \frac{1}{2}\ln\left\{\frac{1+p}{\left(1+p^{1/3}\right)^3}\right\}\right). \quad (19)$$

The three cases are shown in Fig. 5. Note that the power is scaled with respect to the initial internal loss power, $I_0 F_{fr}(t=0)$, hence, a direct comparison of efficiencies is not reasonable from this scaled figure.

5. General cases

The previous discussion pretends that the Ragone curve of an arbitrary dynamic system can be simply derived analytically or numerically, be it of the form of Eq. (1), be it more general. For example, it is in principle straightforward to derive Ragone plots for a hybrid system consisting of two different batteries being parallel, or battery and capacitor, etc. In general, however, things are far more complicated. 'General' means that both potential energy and inductive energy appear. As an illustration, we consider the simple example where the ideal battery is coupled in series to an inductance L (inset Fig. 6). Using the definitions of the previous sections, Eq. (1) can be written as

$$L\dot{I} = -\frac{R}{I}(I - I_+)(I - I_-), \quad (20)$$

where the $I_\pm$ are given in Eq. (2) and are obviously steady states of Eq. (20). Since $I_+ \geq I_- \geq 0$, it turns out that I_+ is linearly stable, while I_- is unstable. Indeed, replacement of I in Eq. (20) by the weakly perturbed steady states, $I_\pm + \Delta I$, leads to an ODE for the perturbation ΔI. Since ΔI is assumed to be small, only the linear terms are considered. The resulting ODE is of the form $\Delta\dot{I} = \lambda_\pm \Delta I$, where $\pm$ indicates the steady state $I_\pm$ under consideration. One readily finds that λ_+ is negative while λ_- is positive, such that perturbations of I_+ and I_- are damped out and increase with time, respectively. A similar behavior is valid for the capacitor. In other words: in the presence of an inductance, the solution needed for the Ragone plot (here: I_-) is not stable and cannot be obtained by using a load with $U = P/I$.

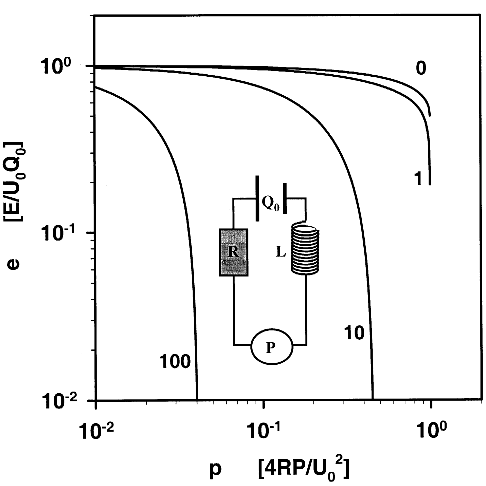

Fig. 6. Normalized Ragone curves of the battery with inductance (inset) for various values of $LU_0/8R^2Q_0$.

This forces us to go back to physics and to the modelling of an element which draws a constant active power. Let us recall: for the ideal battery without inductance, the constant power sink is just a constant resistance (active power sink). In the general case, however, the load could be a general impedance with a certain apparent power. An appropriate reactance could remove the instability. In the following, we sketch how this works for the battery with inductance by an infinitesimal over-compensation of the inductance.

Consider thus a load in Eq. (1) with a voltage $U = P_0/I - L_P\dot{I}$, instead of only $U = P/I$. We do not ask how the load is able to keep this voltage–current relation, but consider it rather as a black box containing probably some complicated power electronics for appropriate control. Furthermore, since we are mathematically faced with a first-order ODE, we have to specify the realistic initial conditions. We suppose that $I(t \to 0) \to 0$. The load power is given by $P = UI = P_0 - (L_P/2)\mathrm{d}I^2/\mathrm{d}t$. Because I is a function of time, P is not constant as required. We take $L_P = L + \delta L$ with δL positive and very small. Hence, L has to be replaced by $-\delta L$ in Eq. (20). Due to the change of the sign of the coefficient of $\dot{I}$, the steady states $I_\pm$ interchange their stability properties: I_- becomes stable. Due to the smallness of δL, the relaxation time which is proportional to δL can be made arbitrarily small. Summarizing, I relaxes very fast to the desired solution I_-. In the limit $\delta L \to 0$, the power is practically constant, $P = P_0$, except in a small initial time interval. Up to $t = t_\infty$, the energy gained by the load is $E(P) = Pt_\infty - LI^2/2$, were we used $I(0) = 0$ and $I(t_\infty) = I(=I_-)$. The time t_∞ is obtained from Eq. (3). We recognize that for constant current I, the energy $LI^2/2$ is stored in the inductance, which reduces the total amount of available energy. However, at $t = t_\infty$, this secondary energy can be recovered by a zero resistance bypass across the battery, and running the whole device as an inductive ESD with initial energy $LI^2/2$. Using the result of Eq. (13), we finally obtain the Ragone curve (primary and secondary energy)

$$E_{bi}(P) = \frac{PQ_0}{I} + \frac{L}{2R}\left(P\ln\left(1 + \frac{RI^2}{P}\right) - RI^2\right). \quad (21)$$

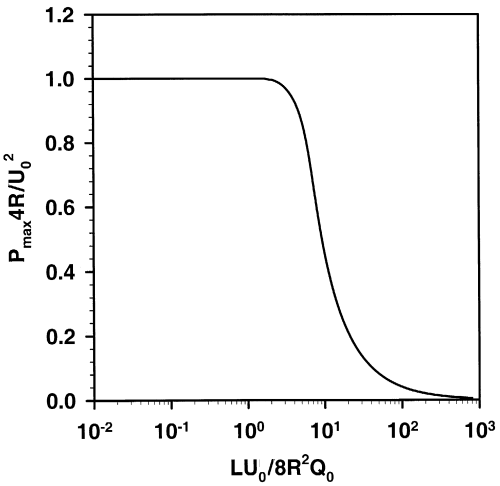

Fig. 7. Maximum available battery power as a function of the normalized inductance.

with $I \equiv I_-(P)$ from Eq. (2). As one expects, if L is small, one has only a weak correction to the Ragone curve of the pure battery; this fact is not mathematically trivial since adding an inductance corresponds to a singular perturbation of a device storing only potential energy. The result of Eq. (21) can be expressed in dimensionless units, $e_{bi} = E_{bi}/U_0Q_0$ and $p = 4RP/U_0^2$.

$$e_{bi}(p) = \frac{1}{2}\frac{p}{1-\sqrt{1-p}} \tag{22}$$

$$+ \frac{LU_0}{8R^2Q_0}\left(p \ln\left(1 + \frac{\left(1-\sqrt{1-p}\right)^2}{p}\right)\right. \tag{23}$$

$$\left. - \left(1-\sqrt{1-p}\right)^2\right), \tag{24}$$

which is shown in Fig. 6 for various values of the inductance in units of $8R^2Q_0/U_0$. In general, the presence of an inductance lowers the available energy and the maximum power of a battery. Moreover, it turns out that for sufficiently large inductance, $E_{bi} = 0$ for $P < U_0^2/4R$. In Fig. 7, the dependence of P_{max} on L is shown in dimensionless units.

6. Conclusion

We introduced a mathematical scheme for the calculation of the Ragone plots of arbitrary energy storage devices. One has to solve the dynamic problem of the circuit of Fig. 2 with a load drawing a constant power, and to determine the time t_∞ when the ESD fails to be able to provide the desired power P. The Ragone curve is then given by $E(P) = Pt_\infty(P)$. It is important that in the case where there is more than one solution branch, $E_1(P)$, $E_2(P)\ldots$, the Ragone plot belongs to the maximum energy, $E(P) = \max_n\{E_n(P)\}$.

In the case of an ESD containing purely kinetic energy or purely potential energy, the power sink can be modelled by a voltage (or a force) equal to $-P/\dot{Q}$, where $\dot{Q}$ is the velocity of the dynamic variable. For the battery and the SMES, additional loss mechanisms as leakage and ohmic bypass, respectively, were included.

It turns out that, for realistic cases, a finite available energy is restricted to a finite power region $0 < P < P_{max}$. For ESD storing potential energy, the high and low power limit is determined by internal friction losses and leakage, respectively. For ESD storing kinetic energy, the low and high power limit is determined by internal friction and bypass losses, respectively. As one result, a consideration of practical Ragone plots allows to draw conclusions for leakage and loss mechanisms. As another result important for engineers, it is possible to figure out bounds for leakage, loss, etc., if one focusses on a specific application.

In the general case of mixed energy devices, the load is a more complicated device, containing reactive parts, in order to keep a constant active power. As an illustrative example, we discussed the battery with series inductance, where the load has to compensate the inductive voltage. For the sake of clearness, the examples discussed in this paper were throughout analytically solvable.

We note that there exist different ways to calculate energy–power relations of ESD. An example is provided by a very convenient linear approach in the case where the information on the frequency dependent impedance of the ESD is given in Ref. [5]. However, we believe the present approach to be the most natural one.

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